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Discrete systems and abelian sandpiles

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ABSTRACT

Computations on the sandpile model lead to questions concerning discrete behaviors. It is shown that the original question has an answer which is not helpful for the study of sandpiles. A variation of the question, using periodic solutions, leads to better answers. These might have an interpretation for the study of sandpiles.

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1. Introduction (after K. Schmidt and E. Verbitskiy)

The sandpile model was introduced by Bak, Tang and Wiesenfeld [3,4] and since then attracted a lot of attention. An even greater interest in this model arose in the mathematics community after the discovery of the abelian property by Dhar [5].

Abelian Sandpile Model (ASM) exhibits the so-called *self-organized criticality*.¹ For an accessible self-contained introduction into the ASM see [9].

There is a number of open problems related to the infinite volume ASM. Most notably, the question on the definition of the sandpile dynamics in infinite volume and a related question on the uniqueness of measure with maximal entropy. Partial positive result have recently been obtained by Athreya and Járai [1,2]. The method is based on mapping the configurations of the infinite volume ASM into the lattice spanning trees, and using the results of Pemantle [8] for spanning trees. The drawback of this method is that the coding map is not *local*.

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¹ See http://en.wikipedia.org/wiki/Self-organized_criticality.

In [10] a different approach has been proposed. Configurations of the infinite volume ASM are encoded as elements of a certain compact abelian group. This group is known to have a unique measure of maximal entropy. If the coding map would be one-to-one, one would immediately conclude that the infinite volume Abelian Sandpile Model has a unique measure of maximal entropy as well. Unfortunately, the coding map is not one-to-one. Nevertheless, if the kernel of the coding map is shown to be *not too big* (e.g., has measure 0 with respect to any measure of maximal entropy), then the uniqueness question could still be answered positively.

It turns out that the kernel of the coding map is closely linked to multidimensional behavior, i.e., the discrete system $\mathcal{A}(n, \mathbb{Z}, **)$ consisting of a certain class of functions $f: \mathbb{Z}^n \rightarrow \mathbb{Z}$ on which the ring of shift operators $D = \mathbb{Z}[u_1, u_1^{-1}, \dots, u_n, u_n^{-1}]$ acts. Understanding the structure of this discrete system is crucial. It is precisely the question on the structure which we address in the next sections.

Before going into the mathematical details, we describe the physical meaning of this model (cf. [10]). Let $\Lambda \subset \mathbb{Z}^2$ be non empty and finite, and that every location/site holds certain grains of sand. Hence a (finite) *configuration* is a function $x: \Lambda \rightarrow \mathbb{N}$. This configuration is *stable* if $x(s) \leq 4$ for all $s \in \Lambda$. If x is *unstable*, i.e., there is a site s such that $x(s) > 4$, then this sandpile topples and the effect is to give 1 grain to each of its neighbours (some grains are lost if s is in the boundary of Λ). It can be shown that the toppling effect is “Abelian,” i.e., independent of the ordering; and any configuration can be stabilized with finite number of topplings. If we write a stabilization of x as $\mathcal{T}(x)$, then a stable configuration x is called *recurrent* when there exists $y: \Lambda \rightarrow \mathbb{N}$ such that $\mathcal{T}(x + y) = x$ here the addition $+$ is defined in the most intuitive sense. Now the infinite volume ASM is the set

$$\mathcal{R}_\infty := \{x: \mathbb{Z}^2 \rightarrow \mathbb{N}, x \text{ stable} \mid x|_\Lambda \text{ is recurrent for all finite subset } \Lambda\}$$

and a *coding map* is from \mathcal{R}_∞ to a fixed compact group X .

2. Discrete systems

The discrete systems that interest us here consist of a ring of operators D (commutative and containing 1), acting upon a certain (additive) space of functions \mathcal{A} , called a *signal space*. In other words, \mathcal{A} is a D -module. We recall that \mathcal{A} is called an *injective cogenerator* if \mathcal{A} is an injective D -module and moreover, for every finitely generated D -module $M \neq 0$ one has $\text{Hom}_D(M, \mathcal{A}) \neq 0$. Further \mathcal{A} is called a *minimal injective cogenerator* if it is an injective cogenerator and no proper D -submodule of it is an injective cogenerator. We refer to Oberst [6,7] for more details. Now we specialize to the cases that are related to the problems that arise for abelian sandpiles.

Let R be a commutative ring having a unit element. Let $\mathcal{A}(n, R, \text{all})$ denote the set of all maps $\mathbb{Z}^n \rightarrow R$. For $i = 1, \dots, n$ one defines the shifts u_i, u_i^{-1} on $\mathcal{A}(n, R, \text{all})$ by $(u_i f)(\dots, m_i, \dots) = f(\dots, m_i + 1, \dots)$ and $(u_i^{-1} f)(\dots, m_i, \dots) = f(\dots, m_i - 1, \dots)$. The ring of operators $D = R[u_1, u_1^{-1}, \dots, u_n, u_n^{-1}]$ acts on $\mathcal{A}(n, R, \text{all})$ and makes this into a module.

It is known that, in case $R = F$ is a field, $\mathcal{A}(n, F, \text{all})$ is an injective cogenerator. It contains a (the) minimal injective cogenerator $\mathcal{A}(n, F, \text{min})$, which can be described by (see Oberst [7, Theorem 1.14]):

f belongs to this minimal object if and only if Df has finite dimension over F or equivalently, the ideal $I = \{d \in D \mid df = 0\}$ satisfies $\dim_F D/I < \infty$.

The following observation seems to be new.

Proposition 2.1. *Suppose that F is a finite field or, more generally, F is algebraic over \mathbb{F}_p for some prime number p . Then $\mathcal{A}(n, F, \text{min})$ consists of the periodic functions $f: \mathbb{Z}^n \rightarrow F$, i.e., the functions which are invariant under some subgroup (lattice) $\Lambda \subset \mathbb{Z}^n$ of finite index.*

Proof. Let the ideal $I \subset D$ satisfy $\dim_F D/I < \infty$. Then I contains, for each $i = 1, \dots, n$, an element of the form $P_i := u_i^d + a_{d-1}u_i^{d-1} + \dots + a_1u_i + a_0$ with all $a_i \in F$ and $a_0 \neq 0$. The zero's of P_i lie in a finite field with, say q elements. Hence a zero of P_i is also a zero of the polynomial $u_i^{q-1} - 1$. The

zero's of P_i can have a certain multiplicity, which we can assume to be $\leq p^m$ for some $m \geq 0$. Then P_i divides the polynomial $(u_i^{q-1} - 1)^{p^m} = u_i^{p^m(q-1)} - 1$. We conclude that for a suitable power q of the characteristic p and a suitable m , the ideal I contains $(u_1^{p^m(q-1)} - 1, \dots, u_n^{p^m(q-1)} - 1)$. An element f with $I \cdot f = 0$ is $p^m(q-1)$ -periodic for 'each direction' in \mathbb{Z}^n . \square

Remark 2.2. If one replaces \mathbb{Z}^n by \mathbb{N}^n , then $\mathcal{A}(n, F, \text{all})$ consists of all maps $f: \mathbb{N}^n \rightarrow F$, where F is a field. This is an injective cogenerator as a module over $D := F[u_1, \dots, u_n]$. The minimal injective cogenerator $\mathcal{A}(n, F, \text{min})$ has been described by Oberst [7, Theorem 1.14]. In case F is a finite field or algebraic over a finite field, the elements of $\mathcal{A}(n, F, \text{min})$ are the functions f with "periodic tails." This means that f is the sum of a function with finite support and a function which has a certain period for each of the n directions of \mathbb{N}^n .

3. The problem

The signal space $\mathcal{A}_\infty := \mathcal{A}(2, \mathbb{Z}, \text{bounded})$, i.e., the set of the bounded functions $x: \mathbb{Z}^2 \rightarrow \mathbb{Z}$, is a module over $D := \mathbb{Z}[u_1, u_1^{-1}, u_2, u_2^{-1}]$. Given are two elements $f, g \in D$ such that $D/(f, g)$ is a free \mathbb{Z} module of finite rank. This condition is somewhat stronger than f, g are relatively prime and it implies that, for every prime number p , the images of f, g in $D/(p) = \mathbb{F}_p[u_1, u_1^{-1}, u_2, u_2^{-1}]$ are $\neq 0$ and not units and are again relatively prime.

One wants to find all solutions $(x, y) \in \mathcal{A}_\infty^2$ of the equation $fx = gy$. The trivial solutions are $\{(gz, fz) \mid z \in \mathcal{A}_\infty\}$ and thus one wants to study the object

$$H_\infty := \frac{\{(x, y) \in \mathcal{A}_\infty^2 \mid fx = gy\}}{\{(gz, fz) \mid z \in \mathcal{A}_\infty\}} \quad \text{and hopes to conclude that it is 'small.'}$$

The condition on f, g implies that

$$0 \leftarrow D/(f, g) \leftarrow D \xleftarrow{d_0} D^2 \xleftarrow{d_1} D \leftarrow 0,$$

with $d_0(a, b) = ag + bf$, $d_1(1) = (f, -g)$, is a free resolution of $D/(f, g)$. Taking $\text{Hom}(-, \mathcal{A}_\infty)$ and forgetting the left term, one obtains the complex K :

$$0 \xrightarrow{d^{-1}} \mathcal{A}_\infty \xrightarrow{d^0} \mathcal{A}_\infty^2 \xrightarrow{d^1} \mathcal{A}_\infty \xrightarrow{d^2} 0, \quad \text{with } d^0(z) = (gz, fz), \quad d^1(x, y) = fx - gy.$$

Then $\text{Ext}^i(D/(f, g), \mathcal{A}_\infty) = \ker d^i / \text{im } d^{i-1}$ for all $i \geq 0$. In particular

$$H_\infty = \text{Ext}^1(D/(f, g), \mathcal{A}_\infty) \quad \text{and} \quad \text{Ext}^2(D/(f, g), \mathcal{A}_\infty) = \frac{\mathcal{A}_\infty}{f\mathcal{A}_\infty + g\mathcal{A}_\infty}.$$

In the above, we may replace \mathcal{A}_∞ by another interesting signal space \mathcal{A}_{per} , consisting of the functions $x: \mathbb{Z}^2 \rightarrow \mathbb{Z}$, which are periodic in the sense that there is a subgroup $\Lambda \subset \mathbb{Z}^2$ of finite index (depending on x) such that x is invariant under Λ . Of course $\mathcal{A}_{\text{per}} \subset \mathcal{A}_\infty$.

Let p be a prime number. Then $\mathcal{A}_\infty/(p)$ coincides with $\mathcal{A}(2, \mathbb{F}_p, \text{all})$, i.e., all functions $x: \mathbb{Z}^2 \rightarrow \mathbb{F}_p$. The latter is an injective cogenerator, as module over $D/(p) = \mathbb{F}_p[u_1, u_1^{-1}, u_2, u_2^{-1}]$. Further, $\mathcal{A}_{\text{per}}/(p)$ coincides with $\mathcal{A}(2, \mathbb{F}_p, \text{per})$, the set of the periodic maps $\mathbb{Z}^2 \rightarrow \mathbb{F}_p$, which is, as $\mathbb{F}_p[u_1, u_1^{-1}, u_2, u_2^{-1}]$ -module, again an injective cogenerator according to Proposition 2.1.

The next step is to consider the exact sequence of complexes

$$0 \rightarrow K \xrightarrow{p} K \rightarrow K_p \rightarrow 0, \quad \text{where } K \text{ is } 0 \xrightarrow{d^{-1}} \mathcal{A}_\infty \xrightarrow{d^0} \mathcal{A}_\infty^2 \xrightarrow{d^1} \mathcal{A}_\infty \rightarrow 0$$

or the same with \mathcal{A}_∞ replaced by \mathcal{A}_{per} and where K_p is the complex K/pK of $D/(p)$ -modules. By assumption the images of f, g in $D/(p)$ are relatively prime and the cohomology groups of the

complex K_p compute, as before, the groups $\text{Ext}_{D/(p)}^i(D/(p, f, g), \mathcal{A}(2, \mathbb{F}_p, \text{all}))$ or in the other case the groups $\text{Ext}_{D/(p)}^i(D/(p, f, g), \mathcal{A}(2, \mathbb{F}_p, \text{per}))$. (Here one uses that the image of f, g in $D/(p)$ are relatively prime.) Since these signal spaces are injective, only the Ext^0 term can be different from 0.

The exact sequence of complexes yields the long exact sequence

$$0 \rightarrow \{z \in \mathcal{A} \mid fz = gz = 0\} \xrightarrow{\cdot p} \{z \in \mathcal{A} \mid fz = gz = 0\} \rightarrow \{z \in \mathcal{A}/(p) \mid fz = gz = 0\} \\ \rightarrow \text{Ext}^1(\bar{D}, \mathcal{A}) \xrightarrow{\cdot p} \text{Ext}^1(\bar{D}, \mathcal{A}) \rightarrow 0 \rightarrow \text{Ext}^2(\bar{D}, \mathcal{A}) \xrightarrow{\cdot p} \text{Ext}^2(\bar{D}, \mathcal{A}) \rightarrow 0.$$

Here $\bar{D} := D/(f, g)$ and \mathcal{A} stands for \mathcal{A}_∞ or \mathcal{A}_{per} . It follows that $\text{Ext}^1(\bar{D}, \mathcal{A})$ is a divisible group and that $\text{Ext}^2(\bar{D}, \mathcal{A})$ is a vector space over \mathbb{Q} . In order to make these cohomology groups explicit we calculate some examples.

Example 3.1. The bounded case: $f = u_2 - 1$, $g = u_1 - 1$, $\mathcal{A} = \mathcal{A}_\infty$.

We observe that $\{z \in \mathcal{A}_\infty \mid (u_1 - 1)z = (u_2 - 1)z = 0\} = \mathbb{Z}$ (i.e., the constant functions with values in \mathbb{Z}) and $\{z \in \mathcal{A}_\infty/(p) \mid (u_1 - 1)z = (u_2 - 1)z = 0\} = \mathbb{F}_p$. Thus also

$$H_\infty = \text{Ext}^1(D/(f, g), \mathcal{A}_\infty) = \frac{\{(x, y) \in \mathcal{A}_\infty^2 \mid (u_2 - 1)x = (u_1 - 1)y\}}{\{((u_1 - 1)z, (u_2 - 1)z) \mid z \in \mathcal{A}_\infty\}}$$

is a \mathbb{Q} -vector space. Consider a pair $(x, y) \in \mathcal{A}_\infty^2$ with $(u_2 - 1)x = (u_1 - 1)y$. There exists a $z \in \mathcal{A}(2, \mathbb{Z}, \text{all})$ with $((u_1 - 1)z, (u_2 - 1)z) = (x, y)$ and this z is unique up to an element in \mathbb{Z} . Then H_∞ can be identified with

$$\frac{\{z \in \mathcal{A}(2, \mathbb{Z}, \text{all}) \mid (u_1 - 1)z, (u_2 - 1)z \in \mathcal{A}_\infty\}}{\{z \in \mathcal{A}_\infty\}}.$$

H_∞ has a structure as \mathbb{R} -vector space. Write \bar{z} for the image in H_∞ of an element z with $(u_1 - 1)z, (u_2 - 1)z \in \mathcal{A}_\infty$. For $\lambda \in \mathbb{R}$, one defines $\lambda\bar{z}$ to be the image in H_∞ of the function $(n, m) \in \mathbb{Z}^2 \mapsto [\lambda z(n, m)]$, where $[\]$ denotes the ‘entier’ function. A straightforward verification shows that this definition makes sense and that it defines on H_∞ the structure of a \mathbb{R} -vector space.

The next step is to show that H_∞ is large, as \mathbb{R} -vector space, by comparing it with the real vector space

$$HC^\infty := \frac{\{f : \mathbb{R}^2 \rightarrow \mathbb{R} \mid f \text{ is } C^\infty \text{ function and } f'_{x_1}, f'_{x_2} \text{ are bounded}\}}{\{f : \mathbb{R}^2 \rightarrow \mathbb{R} \mid f \text{ is } C^\infty \text{ function and all } f, f'_{x_1}, f'_{x_2} \text{ are bounded}\}}.$$

Define the map $\phi : HC^\infty \rightarrow H_\infty$ by $\phi : \bar{f} \mapsto \bar{z}$, where $z(n, m) = [f(n, m)]$. One has the following result.

Proposition 3.2. $\phi : HC^\infty \rightarrow H_\infty$ is an isomorphism of \mathbb{R} -vector spaces.

Proof. We omit the verification that ϕ is well defined and \mathbb{R} -linear. First we show that ϕ is injective.

Suppose that $\phi(\bar{f}) = 0$. Then the function $(n, m) \mapsto f(n, m)$ is bounded. For $(x_1, x_2) \in \mathbb{R}^2$, one has $([x_1], [x_2]) \in \mathbb{Z}^2$ and $\theta_i = x_i - [x_i]$ lies in $[0, 1)$. Let ℓ be the line segment in \mathbb{R}^2 , joining the points (x_1, x_2) and $([x_1], [x_2])$. By the mean value theorem, there exists a (c_1, c_2) on ℓ such that $f(x_1, x_2) = f([x_1], [x_2]) + \theta_1 f'_{x_1}(c_1, c_2) + \theta_2 f'_{x_2}(c_1, c_2)$. Then f is bounded and $\bar{f} = 0$.

The surjectivity of ϕ is a longer exercise that we will only sketch. Let \bar{z} be given. Make a piecewise linear $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\tilde{f}(n, m) = z(n, m)$ for all $(n, m) \in \mathbb{Z}^2$. The set where \tilde{f} is not differentiable are the lines $x_1 \in \mathbb{Z}$ and $x_2 \in \mathbb{Z}$. Outside this set the derivatives $\tilde{f}'_{x_1}, \tilde{f}'_{x_2}$ are bounded. It can be seen that \tilde{f} can be ‘smoothened’, i.e., approximated by an C^∞ -function f , still with bounded derivatives f'_{x_1}, f'_{x_2} . \square

Corollary 3.3. *The real vector space H_∞ has uncountable dimension.*

Proof. For any $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, we define the function $f_\lambda(x_1, x_2) := e^{-\frac{1}{x_1} x_1^\lambda}$ if $x_1, x_2 \geq 0$ and $f_\lambda(x_1, x_2) := 0$ if $x_1 \leq 0$ or $x_2 \leq 0$. The derivatives of f_λ are bounded. Consider any N and $0 < \lambda_1 < \dots < \lambda_N$ and any real linear combination $f := \sum \alpha_i f_{\lambda_i}$ with $\alpha_N \neq 0$. The function f is not bounded because the growth of f is determined by the growth of f_{λ_N} . The corollary now follows from the proposition. \square

We conclude this example by stating that H_∞ is rather large as a real vector space. It is unlikely that H_∞ is ‘small’ in a sense that is useful for the sandpile model, i.e., in showing that the ‘kernel of the coding map’ is not too big with respect to any measure of maximal entropy. \square

Example 3.4. The periodic case: $f = u_2 - 1$, $g = u_1 - 1$, $\mathcal{A} = \mathcal{A}_{\text{per}}$.

Write $\bar{D} = D/(u_1 - 1, u_2 - 1)$. Clearly, one has $\text{Hom}(\bar{D}, \mathcal{A}_{\text{per}}) = \mathbb{Z}$ and $\text{Hom}(\bar{D}/(p), \mathcal{A}_{\text{per}}/(p)) = \mathbb{F}_p$, for any prime number p . Hence $\text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})$ is a vector space over \mathbb{Q} . For any integer $N \geq 1$, we write $\mathcal{A}_{\text{per}, N}$ for the module of the functions $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ which are periodic for the lattice $N\mathbb{Z}^2 \subset \mathbb{Z}^2$. By definition $\mathcal{A}_{\text{per}} = \bigcup_{N \geq 1} \mathcal{A}_{\text{per}, N}$. We will give explicit isomorphisms

$$\begin{aligned} \text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}}) &= \frac{\{(x, y) \in \mathcal{A}_{\text{per}}^2 \mid (u_2 - 1)x = (u_1 - 1)y\}}{\{((u_1 - 1)z, (u_2 - 1)z) \mid z \in \mathcal{A}_{\text{per}}\}} \rightarrow \mathbb{Q}^2 \quad \text{and} \\ \text{Ext}^2(\bar{D}, \mathcal{A}_{\text{per}}) &= \frac{\mathcal{A}_{\text{per}}}{(u_1 - 1)\mathcal{A}_{\text{per}} + (u_2 - 1)\mathcal{A}_{\text{per}}} \rightarrow \mathbb{Q}. \end{aligned}$$

For a fixed integer $N \geq 1$, the groups $\text{Ext}^i(\bar{D}, \mathcal{A}_{\text{per}, N})$ are the cohomology groups of the complex

$$0 \rightarrow \mathcal{A}_{\text{per}, N} \xrightarrow{d^0} \mathcal{A}_{\text{per}, N}^2 \xrightarrow{d^1} \mathcal{A}_{\text{per}, N} \rightarrow 0 \quad \text{with} \\ d^0(z) = ((u_1 - 1)z, (u_2 - 1)z) \quad \text{and} \quad d^1(x, y) = (u_2 - 1)x - (u_1 - 1)y.$$

Consider the maps

$$\begin{aligned} A_N : \ker(d^1) &\rightarrow \left(\frac{1}{N}\mathbb{Z}\right)^2 \quad \text{with } A_N(x, y) = \left(\frac{1}{N} \sum_{i=0}^{N-1} x(i, 0), \frac{1}{N} \sum_{j=0}^{N-1} y(0, j)\right), \\ B_N : \mathcal{A}_{\text{per}, N} &\rightarrow \frac{1}{N^2}\mathbb{Z} \quad \text{with } B_N(z) = \frac{1}{N^2} \sum_{0 \leq i, j < N} z(i, j). \end{aligned}$$

Some straightforward calculations show that A_N and B_N are surjective, that the kernel of A_N is the image of d^0 and that the kernel of B_N is the image of d^1 (this can be seen by reduction modulo prime numbers). Thus $\text{Ext}^i(\bar{D}, \mathcal{A}_{\text{per}, N})$ is isomorphic to \mathbb{Z} , $(\frac{1}{N}\mathbb{Z})^2$, $\frac{1}{N^2}\mathbb{Z}$ for $i = 0, 1, 2$. The isomorphisms are compatible with the inclusions $\mathcal{A}_{\text{per}, N} \subset \mathcal{A}_{\text{per}, M}$ for $N|M$. Taking the direct limit (for the relation $N|M$) over all integers $N \geq 1$, one obtains the required isomorphisms. \square

Now we consider examples which are more closely related to the problem concerning sandpiles. The important operator is the discrete Laplace operator $\Delta := 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1}$. If in the sandpile model there is a point $(n, m) \in \mathbb{Z}^2$ with > 4 grains of sand. Then 4 grains are taken off and one grain of sand is added to each of the 4 neighbours of (n, m) . This is called ‘toppling’. Clearly, the operator Δ is related to toppling. An example of an important pair of operators is $f = \Delta$, $g = (u_1 - 1)^3$. In the following we prefer to use the variables $s_1 = u_1 - 1$, $s_2 = u_2 - 1$.

Define $\tilde{\Delta} := -u_1 u_2 \Delta = s_1^2 + s_2^2 + s_1^2 s_2 + s_1 s_2^2$. The following observation (the proof is straightforward) is handy for computations.

Observation 3.5.

1. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a solution of $(u - 1)^N f = 0$, where u is the usual shift and $N \geq 1$, then f can be presented as $f(n) = \sum_{0 \leq i < N} \alpha(i) \binom{n}{i}$ for all $n \in \mathbb{Z}$ with $\alpha(i) \in \mathbb{Z}$ independent of n .
2. Similarly, if $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ is a solution of $s_1^N f = 0$ and $s_2^N f = 0$, then f is of the form

$$f(n, m) = \sum_{0 \leq i, j < N} \alpha(i, j) \binom{n}{i} \binom{m}{j}$$

for all $(n, m) \in \mathbb{Z}^2$ with $\alpha(i, j) \in \mathbb{Z}$ independent of n, m .

Example 3.6. The bounded case: $f = \tilde{\Delta}$, $g = s_1^3$, $\mathcal{A} = \mathcal{A}_\infty$.

The object to study is $\tilde{H} = \frac{\{(\alpha, y) \in \mathcal{A}_\infty^2 \mid \tilde{\Delta} \alpha = s_1^3 y\}}{\{(s_1^2 z, \tilde{\Delta} z) \mid z \in \mathcal{A}_\infty\}}$, recall the definition

$$H_\infty = \frac{\{z : \mathbb{Z}^2 \rightarrow \mathbb{Z} \mid s_1 z \text{ and } s_2 z \text{ bounded}\}}{\{z : \mathbb{Z}^2 \rightarrow \mathbb{Z} \mid z \text{ bounded}\}}.$$

Our aim is to show that \tilde{H} is again a rather large object, more precisely, it contains a real vector space of uncountable dimension. The idea is simple (but not obvious), we define an additive map $\chi : H_\infty \rightarrow \tilde{H}$ (probably surjective) such that the kernel of χ is isomorphic to the ‘small’ group \mathbb{Z}^2 .

Proposition 3.7. Let $\chi : H_\infty \rightarrow \tilde{H}$ with $\chi(\bar{z}) = \overline{(s_1^3 z, \tilde{\Delta} z)}$, then $\ker \chi \cong \mathbb{Z}^2$.

Proof. The kernel is the set

$$\ker \chi = \{\bar{z} \in H_\infty \mid \exists w : \mathbb{Z}^2 \rightarrow \mathbb{Z} \text{ s.t. } s_1 w, s_2 w \in \mathcal{A}_\infty, \bar{z} = \bar{w} \text{ and } s_1^3 w = \tilde{\Delta} w = 0\}.$$

Let $A_0 := \{w : \mathbb{Z}^2 \rightarrow \mathbb{Z} \mid s_1 w, s_2 w \in \mathcal{A}_\infty, s_1^3 w = \tilde{\Delta} w = 0\}$. Then there is a canonical surjective homomorphism $\varsigma : A_0 \rightarrow \ker \chi$. We compute A_0 and the kernel of ς . If $w \in \mathcal{A}(2, \mathbb{Z}, \text{all})$ satisfies $s_1^3 w = \tilde{\Delta} w = 0$, then by Observation 3.5 one has

$$w(n, m) = \sum_{0 \leq j < 4} \alpha(0, j) \binom{n}{0} \binom{m}{j} + \sum_{0 \leq j < 2} \alpha(1, j) \binom{n}{1} \binom{m}{j} + \sum_{0 \leq j < 2} \alpha(2, j) \binom{n}{2} \binom{m}{j}$$

for all (n, m) and the coefficients $\alpha(\dots)$ have the relations $\alpha(0, 3) + \alpha(2, 1) = 0$ and $\alpha(0, 2) + \alpha(2, 0) + \alpha(2, 1) = 0$. Moreover, since $s_1 w, s_2 w$ are bounded, one has $w(n, m) = \alpha(0, 0) + \alpha(0, 1) \binom{m}{1} + \alpha(1, 0) \binom{n}{1}$. Hence $A_0 \cong \mathbb{Z}^3$. The kernel of ς is $\{w \in A_0 \mid w \text{ is bounded}\} = \mathbb{Z}$, and therefore $\ker \chi \cong \mathbb{Z}^2$. $\square \quad \square$

Example 3.8. The periodic case: $f = \tilde{\Delta}$, $g = s_1^3$, $\mathcal{A} = \mathcal{A}_{\text{per}}$.

Write $\bar{D} = D/(\tilde{\Delta}, s_1^3)$. The results are:

Proposition 3.9.

$$\text{Ext}^i(\bar{D}, \mathcal{A}_{\text{per}}) \text{ equals } \mathbb{Z} \text{ for } i = 0, \quad (\mathbb{Q}/\mathbb{Z})^5 \oplus \mathbb{Q}^2 \text{ for } i = 1 \text{ and } \mathbb{Q} \text{ for } i = 2.$$

Proof. The ideal $(\tilde{\Delta}, s_1^3)$ is contained in (s_1, s_2) and contains (s_1^3, s_2^4) . Further \bar{D} is seen to be the free \mathbb{Z} -module of rank 6 on generators which are the images of the elements $1, s_1, s_1^2, s_2, s_1 s_2, s_1^2 s_2$. Using Observation 3.5, one computes that $\text{Hom}(\bar{D}, \mathcal{A}(2, \mathbb{Z}, \text{all}))$, which is the set $\{z: \mathbb{Z}^2 \rightarrow \mathbb{Z} \mid \tilde{\Delta} z = s_1^3 z = 0\}$, is the free \mathbb{Z} -module generated by the following six elements

$$\begin{aligned} f_1 &= \binom{n}{0} \binom{m}{0}, & f_2 &= \binom{n}{0} \binom{m}{1}, & f_3 &= \binom{n}{1} \binom{m}{0}, & f_4 &= \binom{n}{1} \binom{m}{1}, \\ f_5 &= \binom{n}{0} \binom{m}{2} - \binom{n}{2} \binom{m}{0}, & f_6 &= \binom{n}{0} \binom{m}{2} + \binom{n}{0} \binom{m}{3} - \binom{n}{2} \binom{m}{1}. \end{aligned}$$

In particular, $\text{Hom}(\bar{D}, \mathcal{A}_{\text{per}}) = \mathbb{Z} f_1 \cong \mathbb{Z}$.

For any integer $N > 1$ the functions $(n, m) \mapsto \binom{n}{i} \cdot \binom{m}{j} \bmod N$ are periodic with respect to the lattice $N\mathbb{Z}^2 \subset \mathbb{Z}^2$. Therefore $\text{Hom}(\bar{D}, \mathcal{A}_{\text{per}}/(N))$ is a free $\mathbb{Z}/N\mathbb{Z}$ -module of rank 6 with, as basis, the images modulo N of the f_1, \dots, f_6 . Further $\text{Ext}^i(\bar{D}, \mathcal{A}_{\text{per}}/(N)) = 0$ for $i > 0$. This follows by reducing this statement to the case where N is a prime number.

Let K be, as before, the complex $0 \rightarrow \mathcal{A}_{\text{per}} \rightarrow \mathcal{A}_{\text{per}}^2 \rightarrow \mathcal{A}_{\text{per}} \rightarrow 0$. The cohomology groups of K are the groups $\text{Ext}^i(\bar{D}, \mathcal{A}_{\text{per}})$ that we want to compute. The exact sequence of complexes $0 \rightarrow K \xrightarrow{N \cdot} K \rightarrow K_N \rightarrow 0$, where $K_N = K/(N)$, yield, as before, a long exact sequence

$$\begin{aligned} 0 \rightarrow \mathbb{Z} f_1 \xrightarrow{N \cdot} \mathbb{Z} f_1 \rightarrow \bigoplus_{i=1}^6 \frac{1}{N} \mathbb{Z}/\mathbb{Z} \cdot f_i \rightarrow \text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}}) \xrightarrow{N \cdot} \text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}}) \\ \rightarrow 0 \rightarrow \text{Ext}^2(\bar{D}, \mathcal{A}_{\text{per}}) \xrightarrow{N \cdot} \text{Ext}^2(\bar{D}, \mathcal{A}_{\text{per}}) \rightarrow 0. \end{aligned}$$

Let $\text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})[N]$ denote the N -torsion part of $\text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})$, i.e., the kernel of $N \cdot$ on this group. Then we have a compatible system of isomorphisms $\bigoplus_{i=2}^6 \frac{1}{N} \mathbb{Z}/\mathbb{Z} \cdot f_i \rightarrow \text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})[N]$. This induces an isomorphism $\bigoplus_{i=2}^6 \mathbb{Q}/\mathbb{Z} \cdot f_i \rightarrow \text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})_{\text{tors}} := \bigcup_{N>1} \text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})[N]$.

The non torsion part of $\text{Ext}^1(\bar{D}, \mathcal{A}_{\text{per}})$ and $\text{Ext}^2(\bar{D}, \mathcal{A}_{\text{per}})$ are vector spaces over \mathbb{Q} . We will calculate these by tensoring with \mathbb{C} (first to remove the torsion). Therefore we consider the spaces $A(N) := \mathcal{A}_{\text{per}, N} \otimes_{\mathbb{Z}} \mathbb{C}$ and the complex

$$0 \rightarrow A(N) \xrightarrow{d^0} A(N)^2 \xrightarrow{d^1} A(N) \rightarrow 0, \quad d^0(z) = (s_1^3 z, \tilde{\Delta} z), \quad d^1(x, y) = \tilde{\Delta} x - s_1^3 y.$$

The complex vector space $A(N)$ of dimension N^2 can be identified with the vector space of all maps $(\mathbb{Z}/N\mathbb{Z})^2 \rightarrow \mathbb{C}$. We use now some Fourier theory. Put $\zeta := e^{2\pi i/N}$. Now $A(N)$ has as basis the characters $\{\chi_{a,b} \mid 0 \leq a, b < N\}$, where $\chi_{a,b}$ is defined by $\chi_{a,b}(n, m) = \zeta^{an+bm}$ for all (n, m) . One observes that

$$\begin{aligned} d^0(\chi_{a,b}) &= ((\zeta^a - 1)^3 \chi_{a,b}, ((\zeta^a - 1)^2 \zeta^b + (\zeta^b - 1)^2 \zeta^a) \chi_{a,b}) \quad \text{and} \\ d^1(\chi_{a,b}, \chi_{c,d}) &= ((\zeta^a - 1)^2 \zeta^b + (\zeta^b - 1)^2 \zeta^a) \chi_{a,b} - (\zeta^c - 1)^3 \chi_{c,d}. \end{aligned}$$

Using this one easily computes that the cohomology groups of the above complex have dimension 1, 2, 1. Further the inclusion $A(N) \subset A(M)$ for $N|M$ induces isomorphisms for the cohomology groups. This proves the required result. \square

The dimensions 1, 2, 1 for the groups $\text{Ext}^i(\bar{D}, \mathcal{A}_{\text{per}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $i = 0, 1, 2$ are maybe explained by observation that the space of periodic functions $\mathcal{A}_{\text{per}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a direct limit of functions on ‘two-dimensional discrete tori’ $(\mathbb{Z}/N\mathbb{Z})^2$. \square

Example 3.10. The general periodic case.

$\mathcal{A} = \mathcal{A}_{per}$ and M is a finitely generated D -module with the properties: $((u_1 - 1)^n, (u_2 - 1)^n)M = 0$ for some $n > 0$ and M is a free finitely generated \mathbb{Z} -module. A case of special interest is $M = D/(\Delta, (u_1 - 1, u_2 - 1)^3)$.

Due to the following observation, the computations used in Example 3.8 will also work in this example:

Proposition 3.11. M has homological dimension ≤ 2 .

Proof. The finitely generated module M over D is supposed to be free and of finite rank as \mathbb{Z} -module. We claim that the homological (or projective) dimension of M is ≤ 2 . It suffices to prove that for every maximal ideal \underline{m} of D , the localization $M_{\underline{m}}$ has, as module over $D_{\underline{m}}$, homological dimension ≤ 2 . The ideal \underline{m} contains a prime number p and p is not a zero divisor on $M_{\underline{m}}$. Therefore the homological codimension (or depth) of $M_{\underline{m}}$ (see [11, pp. 78, 80]) is ≥ 1 . Using [11, Proposition 21, p. 101] and that $D_{\underline{m}}$ is a regular local ring of dimension 3, one obtains that $M_{\underline{m}}$ has homological dimension ≤ 2 . \square

One wants to compute the groups $\text{Ext}^i(M, \mathcal{A}_{per})$. For any prime number p , the groups $\text{Ext}^i(M/(p), \mathcal{A}_{per}/(p))$ are zero for $i > 0$ and for $i = 0$ it is a vector space over \mathbb{F}_p of dimension equal to $\dim M/(p)$, which is the rank of M as \mathbb{Z} -module. Further $\text{Ext}^0(M, \mathcal{A}_{per}) = \text{Hom}(M, \mathcal{A}_{per})$ is a free \mathbb{Z} -module of rank \leq the rank of M . The method of the proof of Proposition 3.9 yields that the torsion part of $\text{Ext}^1(M, \mathcal{A}_{per})$ is equal to $(\mathbb{Q}/\mathbb{Z})^d$ where d is the difference between the rank of M and the rank of $\text{Ext}^0(M, \mathcal{A}_{per})$.

For the computation of the non torsion part of $\text{Ext}^1(M, \mathcal{A}_{per})$ and the \mathbb{Q} -vector space $\text{Ext}^2(M, \mathcal{A}_{per})$ one computes a free resolution

$$0 \leftarrow M \leftarrow D^{a_0} \leftarrow D^{a_1} \leftarrow D^{a_2} \leftarrow 0.$$

We observe that $a_1 = a_0 + a_2$. The essential computation concerns the complex, obtained by replacing \mathcal{A}_{per} by $A(N) = \mathcal{A}_{per, N} \otimes \mathbb{C}$. This complex has the form $0 \rightarrow A(N)^{a_0} \rightarrow A(N)^{a_1} \rightarrow A(N)^{a_2} \rightarrow 0$. It follows that $\sum (-1)^i \dim_{\mathbb{Q}}(\text{Ext}^i(M, \mathcal{A}_{per}) \otimes \mathbb{Q}) = 0$.

We make the above explicit for the example $M = D/I$ where I is the ideal $(\Delta, (u_1 - 1, u_2 - 1)^3) = (s_1^2 + s_2^2, s_1^3, s_2^3)$. Then M is a free \mathbb{Z} -module with rank 5. The free resolution of M is

$$\begin{aligned} 0 \leftarrow M \leftarrow D \leftarrow D^3 \leftarrow D^2 \leftarrow 0, \quad \text{given by} \\ D^3 \rightarrow D : (1, 0, 0), (0, 1, 0), (0, 0, 1) \mapsto s_1^2 + s_2^2, s_1^3, s_2^3 \quad \text{and} \\ D^2 \rightarrow D^3 : (1, 0), (0, 1) \mapsto (-s_1^2 + s_2^2, s_1, -s_2), (s_1 s_2, -s_2, -s_1). \end{aligned}$$

One computes that $\text{Ext}^0(M, \mathcal{A}_{per}) = \mathbb{Z}$, $\text{Ext}^1(M, \mathcal{A}_{per}) = (\mathbb{Q}/\mathbb{Z})^4 \oplus \mathbb{Q}^2$ and $\text{Ext}^2(M, \mathcal{A}_{per}) = \mathbb{Q}$. \square

An analogy. Let \mathbf{B} denote the space of the (real) C^∞ -functions on \mathbb{R}^2 , which are double periodic (both periods 1). On this space one considers the ring of operators $\mathbb{R}[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}]$. We observe that the $U_i := 1 + \frac{\partial}{\partial x_i}$ are invertible operators on \mathbf{B} . Thus we might also consider the ring of differential operators $\mathbf{D} := \mathbb{R}[U_1, U_1^{-1}, U_2, U_2^{-1}]$. This ring is formally isomorphic to $\mathbb{R}[u_1, u_1^{-1}, u_2, u_2^{-1}]$. This isomorphism sends the finitely generated modules M of Example 3.10 to modules \tilde{M} over \mathbf{D} . A natural guess is that the $\text{Ext}^i(M, \mathcal{A}_{per}) \otimes \mathbb{R}$ are functorially isomorphic to the $\text{Ext}^i(\tilde{M}, \mathbf{B})$.

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